THE COMPARISON OF SHEAF-SOLUTIONS IN FUZZY CONTROL PROBLEM

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ABSTRACT: In [2] the author considered the Sheaf-Optimal Control Problem (SOCP) by differential equations:
\[ \frac{dx(t)}{dt} = f(t, x(t), u(t)), \]
where \( x_0 \in Q \subset R^n, \ u \in U \subset R^p, \ t \in [0, T] \subset R^+ \), and sheaf of solutions:
\[ H_{t,u} = \{ x(t) = x(t,x_0,u(t)) | x_0 \in H_0 \subset Q, t \in I = [0, T] \subset R^+, u(t) \in U \} \]
with the goal function \( I(u) \rightarrow \min \).

In [5], we have offered the necessary conditions of Sheaf-Optimal Control Problem in Fuzzy type (SOFCP), that means the controls \( u(t) \in U \subset E^p \) not belong to \( R^p \).

This paper shows some comparison of sheaf-solutions \( H_{t,u} \) and \( H_{t,\pi} \) for many kinds of fuzzy controls \( u(t), \overline{u}(t) \in U \subset E^p \) in Sheaf Fuzzy Control Problem (SFCP)

Keywords: Fuzzy Theory, Optimal Control Theory, Differential Equations.

1. INTRODUCTION:

For Sheaf-Optimal Control Problem (SOCP) many controls \( u(t) \) and \( \overline{u}(t) = u(t) + \Delta u \) are considered with \( \|\Delta u\| = \|\pi(t) - u(t)\| \leq \delta \), where \( u(t), \overline{u}(t) \in U \subset R^p \) [2]. For Sheaf-Optimal Control Problem in Fuzzy Type (SOFCP) we have fuzzy controls \( u(t) \) and \( \overline{u}(t) \in U \subset E^p \) with \( \|\overline{u}(t) - u(t)\| \leq T \sqrt{p} \) [5].

For the Sheaf Fuzzy Control Problem (SFCP) we have the same fuzzy controls \( u(t) \) and \( \overline{u}(t) \in U \subset E^p \), that was defined by definition 5 in [5]. The paper is organized as follows:

In the second section, offering the Sheaf Fuzzy Control Problem (SFCP) we get estimations of the norms \( \|\bullet\| \) and \( \|\bullet\| \) of
\[ \Delta x = x(t,x_0,\overline{u}(t)) - x(t,x_0,u(t)) \]
\[ \Delta f = f(t,x(t,x_0,\overline{u}(t)),\overline{u}(t)) - f(t,x(t,x_0,u(t)),u(t)) \]

In section 3, we study some comparisons of sheaf solutions \( H_{t,u} \) in many kinds of fuzzy controls \( u(t), \overline{u}(t) \in U \subset E^p \), that means we have to compare the measure \( \|\mu(H_{t,u}) - \mu(H_{t,u})\| \)

2. THE SHEAF FUZZY CONTROL PROBLEM (SFCP)

As we know, the solutions of differential equations depend locally on initial, right hand side and parameters. Now, we consider a control system of differential equations
\[ \frac{dx(t)}{dt} = f(t,x(t),u(t)) \quad (1) \]
where \( x(0) = x_0 \in H_0 \subset Q \subset R^n, x(t) \in Q \subset R^n, u(t) \in U \subset E^p, \ t \in I = [0, T] \subset R^+ \) and \( f : I \times R^n \times E^p \rightarrow R^n \).
Definition 1. The sheaf - solution (or sheaf-trajectory) \( x(t, x_0, u) \) which gives at the time \( t \) a set
\[
H_{t,u} = \left\{ x(t) = x(t, x_0, u) | x_0 \in H_0 \subset Q, x(t) - \text{solution of (1)} \right\},
\]
where \( x_0 \in H_0 \subset Q \subset \mathbb{R}^n \), \( u(t) \in U \subset \mathbb{R}^p \), \( t \in I \).

In the case, when a control \( u(t) \) is fuzzy, we have Sheaf Fuzzy Control Problem (SFCP).

Suppose at time \( t = 0, u(0) = 0 \) and \( x(0) = x_0 \in H_0 \). For two admissible controls \( u(t) \) and \( \overline{u}(t) \) \( U \subset \mathbb{R}^p \), we have two sets of sheaf-solutions
\[
H_{t,u} = \left\{ x(t) = x(t, x_0, u) | x_0 \in H_0 \subset Q, x(t) - \text{a solution of (1) by control } u(t) \right\}
\]
\[
H_{t,\overline{u}} = \left\{ \overline{x}(t) = x(t, x_0, \overline{u}(t)) | x_0 \in H_0 \subset Q, \overline{x}(t) - \text{a solution of (1) by control } \overline{u}(t) \right\},
\]
where \( t \in I \). (See fig.1)

Fig. 1. The sheaf-solutions of Sheaf Fuzzy Control Problem (SFCP).

If \( \mu(H_{t,u}) \) is a measure of the set \( H_{t,u} \) then \( \mu(H_{t,u}) \) is called a cross-area of sheaf trajectory at \( (t,u) \), in particular it is a square of set \( H_{t,u} \). That is \( \mu(H_{t,u}) = \int_{H_{t,u}} dx \), and
\[
\mu(H_{t,\overline{u}}) = \int_{H_{t,\overline{u}}} d\overline{x} \text{ is a square of } H_{t,\overline{u}}.
\]

Assumption 1. Suppose that the vector function \( f(t, x(t), u(t)) \) satisfies
i) \[
\left\| \frac{\partial f}{\partial x} \Delta x(t) + \frac{\partial f}{\partial u} \Delta u(t) \right\| \leq M(\|\Delta x(t)\| + \|\Delta u(t)\|)
\]
(3)

ii) \[
\sum_{k=2}^{\infty} \frac{1}{k!} \left\| d^k f \right\| \leq m
\]
(4)

iii) \[
\left\| \text{sp} \frac{\partial f}{\partial x} (t, x(t, x_0, u(t)), u(t)) \right\| = L(\|u(t)\|)
\]
(5)

for all \( x(t) \in Q \subset \mathbb{R}^n \), \( u(t), \overline{u}(t) \in U \subset \mathbb{R}^p \), \( t \in I \), where \( M, m, L \) are real positive constants and \( \text{sp}A \) is trace of matrix \( A \).

Lemma 1. For the fuzzy controls \( u(t) \) and \( \overline{u}(t) \in U \subset \mathbb{R}^p \), the norm of \( \Delta u = \overline{u}(t) - u(t) \) is estimated as follows:

\[
\begin{align*}
\text{a)} & \quad \| \Delta u \|_{C} \leq \sqrt{p} \\
\text{b)} & \quad \| \Delta u \|_{L} = \int_{0}^{T} \| \Delta u(t) \| \, dt \leq T \sqrt{p},
\end{align*}
\]
(6) (7)
Proof of Lemma 1: Let \( u(t), \bar{u}(t) \in U \subset E^p \) are fuzzy controls. In [5], we defined a fuzzy function \( u: I \to U \subset E^p = E \times E \times \ldots \times E \), that means \( u(t) = (u_1(t), u_2(t), \ldots, u_p(t)) \). Because every \( u_k(t) \) satisfies \( |u_k(t)| \leq 1 \) (\( k = 1, 2, \ldots, p \)) then a norm of

\[
\|Au\|_c = \max \left\{ \left\| \bar{u}(t) - u(t) \right\| : t \in I \right\}
\leq \max \left\{ \sqrt{\sum_{i=1}^{p} (\bar{u}_i(t) - u_i(t))^2} : t \in I \right\} \leq \sqrt{p}
\]

where \( u(t), \bar{u}(t) \in U \subset E^p \).

Proof of Theorem 1: Let \( u(t), \bar{u}(t) \in U \subset E^p \) are fuzzy controls with \( \Delta u = \bar{u}(t) - u(t) \) satisfies (6) or (7).

\[\Delta = \Delta \leq + C x(t, x_0, u(t)) \exp(MT) \quad (8)\]

\[\Delta \leq + \frac{1}{2} L x(t, x_0, u(t)) \exp(MT) \quad (9)\]

Proof of Theorem 1: Let \( u(t), \bar{u}(t) \in U \subset E^p \) are fuzzy controls with \( \Delta u = \bar{u}(t) - u(t) \) satisfies (6) or (7).

a) The solutions of (1) are equivalent the following integrals:

\[
x(t) = x_0 + \int_0^t f(s, x(s), u(s)) \, ds \quad \text{and} \quad \bar{x}(t) = x_0 + \int_0^t f(s, \bar{x}(s), \bar{u}(s)) \, ds.
\]

Estimating \( \|\Delta x(t)\| \) as follows \( \|\Delta x(t)\| \leq \int_0^t \| f(s, \bar{x}(s), \bar{u}(s)) - f(s, x(s), u(s)) \| \, ds \)

\[
\leq \int_0^t \left| \frac{\partial f}{\partial x}(s, x(s), u(s)) \right| dx + \left| \frac{\partial f}{\partial u}(s, x(s), u(s)) \right| du + \sum_{k=2}^p \left| \frac{\partial f}{\partial u_k}(s, x(s), u(s)) \right| du \, ds
\]

\[
\leq M \int_0^t \| dx + du + \ldots \| \, ds \leq M \int_0^t \| \Delta x(s) \| \, ds + M \int_0^t \| \Delta u(s) \| \, ds + M T
\]

\[
\leq M \int_0^t \| \Delta x(s) \| \, ds + M T \sqrt{p} + m T
\]

By Gronwall-Bellmann’s Lemma, it implies that

\[
\|\Delta x\|_c = \max_{t \in [0, T]} \|\Delta x(t)\| \leq T(m + M \sqrt{p}) \exp(MT)
\]

b) \( \|\Delta x(t)\| \leq M \int_0^t \|\Delta x(s)\| \, ds + M \int_0^t \|\Delta u(s)\| \, ds + M T
\]

\[
\|\Delta x(t)\| \leq M \int_0^t \|\Delta x(s)\| \, ds + M T \sqrt{p} + m T
\]

\[
\leq T(m + M \sqrt{p}) \exp(MT)
\]

For \( \|\Delta x\|_L = \int_0^T \|\Delta x(t)\| dt \leq T^2 (M \sqrt{p} + m) \exp(MT) \) we have (9)
Theorem 2. Suppose that \( u(t), \overline{u}(t) \in U \subset E^p \) are fuzzy controls, if the function \( f(t, x(t), u(t)) \) satisfies (3) and (4) then the norm of
\[
\Delta f = f(t, x(t, x_0, \overline{u}(t), \overline{u}(t)) - f(t, x(t, x_0, u(t)), u(t))
\]
is estimated as follows:

(a) \( \|\Delta f\|_\infty \leq MT[(M\sqrt{p} + m) \exp(MT) + \sqrt{p}] + m \) \tag{10}

(b) \( \|\Delta f\|_1 \leq T \left\{ M \left[ T(m + M\sqrt{p}) \exp(MT) + \sqrt{p} \right] + m \right\} \) \tag{11}

Proof of Theorem 2:

(a) For \( \Delta f = \max \left\{ \|df + \frac{1}{2!}d^2f + \frac{1}{3!}d^3f + \ldots : t \in I \right\} \)
\[
\leq \max \left\{ \|df\| + \sum_{k=2}^{\infty} \frac{1}{k!} \|d^k f\| : t \in I \right\}
\]
\[
\leq \max \left\{ \|\frac{\partial f}{\partial x}\| \|dx\| + \frac{\partial f}{\partial u}\| \|du\| + \sum_{k=2}^{\infty} \frac{1}{k!} \|d^k f\| : t \in I \right\}
\]
\[
\leq M (\|\Delta x\|_\infty + \|\Delta u\|_\infty) + m
\]
\[
\leq M[T(M\sqrt{p} + m) \exp(MT) + T\sqrt{p}] + m
\]
\[
\leq MT[(M\sqrt{p} + m) \exp(MT) + \sqrt{p}] + m
\]

(b) For \( \|\Delta f\|_1 = \int_0^T \|f(s, x(s, x_0, \overline{u}(s)), \overline{u}(s)) - f(s, x(s, x_0, u(s)), u(s))\| ds \)
\[
\leq M \left( \int_0^T \|\Delta x(t)\| dt + \int_0^T \|\Delta u(t)\| dt \right) + m \int_0^T dt
\]
\[
\leq M (\|\Delta x\|_\infty + \|\Delta u\|_\infty) + mt
\]
\[
\leq M \left[ T^2 (m + M\sqrt{p}) \exp(MT) + T\sqrt{p} \right] + mT
\]
\[
\leq T \left\{ M \left[ T(m + M\sqrt{p}) \exp(MT) + \sqrt{p} \right] + m \right\}
\]

3. THE COMPARISON OF SHEAF SOLUTIONS IN THE SFCP

Lemma 2. For \( A, B \geq 0 \) there exists a real number \( K \) such that \( e^A - e^B \leq Ke^{A-B} \).

Proof of Lemma 2: We have \( e^A - e^B = e^B(e^{A-B} - 1) \leq Ke^{A-B}, K > e^B \).

Now, suppose that \( \mu(H_0) \) is given. There are many following results of comparison of sheaf-solutions:

Theorem 3. Suppose that \( u(t), \overline{u}(t) \in U \subset E^p \) are fuzzy controls. If the function \( f(t, x(t), u(t)) \) satisfies (3), (4) and (5) then we have the following estimation:
\[
|\mu(H_{t,\pi}) - \mu(H_{t,u})| \leq \mu(H_0) \exp(\text{LT}\sqrt{p})
\]

Proof of Theorem 3: We have \( \mu(H_{t,u}) = \int_{H_{t,u}} dx_t = \int_{H_0} \det \frac{\partial x(t, x_0, u)}{\partial x_0} dx_0 \),

where
that means
\[ \| \Delta f \| = \max \left\{ \left\| f(t,x(t,x_0,u(t)),\overline{u}(t)) - f(t,x(t,x_0,u(t)),u(t)) \right\| : t \in I \right\} \]
\[ \mu(H_{t,u}) = \int_{H_0} \frac{\partial x(t,x_0,u)}{\partial x_0} dx_0 = \int_{H_0} \exp \left( \int_{0}^{T} \frac{\partial f(y,x_0,x_0,u(y))}{\partial x} dy \right) dx_0 \]
\[ \mu(H_{t,u}) = \mu(H_0) \exp(L \int_{0}^{T} \| u(t) \| dt) \].

It is analogous of proof a) above, we have \[ \mu(H_{T,u}) = \mu(H_0) \exp(L \int_{0}^{T} \| \overline{u}(t) \| dt) \].

Estimating \[ | \mu(H_{T,u}) - \mu(H_{T,u}) | \] we have
\[ | \mu(H_{T,u}) - \mu(H_{T,u}) | \leq \mu(H_0) \left[ \exp(L \int_{0}^{T} \| \overline{u}(t) \| dt) - \exp(L \int_{0}^{T} \| u(t) \| dt) \right] \]
\[ \leq \mu(H_0) K \exp \left[ L \int_{0}^{T} \left( \| \overline{u}(t) \| - \| u(t) \| \right) dt \right] \]
\[ \leq \mu(H_0) K \exp \left[ L \int_{0}^{T} \| U(t) \| dt \right] \leq \mu(H_0) K \exp\left[ LT \sqrt{p} \right] \]

where \[ K \geq \exp(LT \sqrt{p}) \].

**Corollary 1**  Suppose that \( u(t), \overline{u}(t) \in U \subset E^p \) are fuzzy controls. If the function \( f(t,x(t),u(t)) \) satisfies (3) and (4), then for (1) when \( n=1 \) we have the following estimation:
\[ | \mu(H_{T,u}) - \mu(H_{T,u}) | \leq (b_0 - a_0) \exp(2LT \sqrt{p}) \),

(13)

where \( K = \exp(LT \sqrt{p}) \).

**Proof of Corollary:** When \( n=1 \) we have \( \mu(H_0) = b_0 - a_0 \), finally we get (13) (see fig. 2).

**Fig. 2.** The sheaf-solutions of Sheaf Fuzzy Control Problem (SFCP), when \( n = 1 \). (■)

**4. CONCLUSION**

In the Sheaf Fuzzy Control Problem (SFCP) for many different fuzzy controls \( u(t), \overline{u}(t) \in U \subset E^p \) we have the comparison (7)-(13). There are differences between the Sheaf Fuzzy Control Problem (SFCP) and the Sheaf Optimal Control Problem in Fuzzy Type (SOFCP) what was offered in [5].

**5. ACKNOWLEDGMENT**
SO SÁNH BỌ NGHỊỆM TRONG BÀI TOÁN ĐIỀU KHIỆN MỎ

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TÓM TÁT: Trong [2] tác giả đã xét bài toán điều khiển tối ưu bó (SOCP) cho bởi hệ phương trình vi phân:

\[ \frac{dx(t)}{dt} = f(t, x(t), u(t)) \]

ở đây \( x_0 \in Q \subset \mathbb{R}^n \), \( u \in U \subset \mathbb{R}^p \), \( t \in [0, T] \subset \mathbb{R}^+ \), và bó nghiệm:

\[ H_{t,u} = \{ x(t) = x(t, x_0, u(t)) | x_0 \in H_0 \subset Q, t \in I = [0, T], u(t) \in U \} \]

với hàm mục tiêu \( I(u) \rightarrow \min \).

Trong [5] lại trình bày các điều kiện cần của bài toán điều khiển tối ưu bó dạng mờ (SOFCP), với các điều kiện mờ \( u(t) \in U \subset \mathbb{E}^p \) thay vì thuộc \( \mathbb{R}^p \).

Bài báo này đưa ra các so sánh các bó nghiệm \( H_{t,u} \) và \( H_{t,v} \) ứng với các điều kiện mờ khác nhau \( u(t), v(t) \in U \subset \mathbb{E}^p \) của bài toán điều khiển bó dạng mờ (SFPC).

Từ khóa: Lý thuyết mờ, Lý thuyết điều khiển tối ưu, Phương trình Vi phân

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